Progress in the prediction of g-2 of the muon

F. Jegerlehner

Institut Fizyki, Uniwersytet Śląski, Katowice
DESY Zeuthen/Humboldt-Universität zu Berlin

Matter to the Deepest, Ustroń, Poland, September 15, 2009

supported by the Alexander von Humboldt Foundation through the Polish Science Foundation
Outline of Talk:

① The Anomalous Magnetic Moment of the Muon: Status

② Improvements due to the new \( \pi^+ \pi^- \) data from BaBar

③ About the hadronic light-by-light scattering contribution

④ Summary and Outlook
The Anomalous Magnetic Moment of the Muon: Status

\[ \vec{\mu} = g_\mu \frac{e \hbar}{2m_\mu c} \vec{s} ; \quad g_\mu = 2 (1 + a_\mu) \]

**Dirac:** \( g_\mu = 2 \), \( a_\mu \) muon anomaly

\[ \gamma(q) \]

\[ \mu(p') \]

\[ \mu(p) \]

\[ = (-ie) \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m_\mu} F_2(q^2) \right] u(p) \]

\[ F_1(0) = 1 ; \quad F_2(0) = a_\mu \]

\( a_\mu \) responsible for the Larmor precession

directly proportional at magic energy \( \sim 3.1 \text{ GeV} \)

CERN, BNL g-2 experiments

\[ \vec{\omega}_a = \frac{e}{m} \left[ a_\mu \vec{B} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \vec{\beta} \times \vec{E} \right]^{E \sim 3.1 \text{ GeV}} \]

at "magic \( \gamma \)"

\[ \simeq \frac{e}{m} \left[ a_\mu \vec{B} \right] \]
Standard Model Prediction for $a_\mu$

QED Contribution

The QED contribution to $a_\mu$ has been computed (or estimated) through 5 loops.

Growing coefficients in the $\alpha/\pi$ expansion reflect the presence of large $\ln \frac{m_\mu}{m_e} \approx 5.3$ terms coming from electron loops.

New: $a_e^{\text{exp}} = 0.001\,159\,652\,180\,73(28)$ Gabrielse et al. 2008

$$\alpha^{-1}(a_e) = 137.035999084(51)[0.37\text{ppb}]$$

based on work of Kinoshita, Nio 04

$$a_\mu^{\text{QED}} = 116\,584\,718.104\, (0.043) \underbrace{\left( \frac{0.014}{\alpha_{\text{inp}}} \right)}_{m_e/m_\mu} \underbrace{\left( \frac{0.025}{\alpha^4} \right)}_{\alpha^4} \underbrace{\left( \frac{0.137}{\alpha^5} \right)}_{\alpha^5} \times 10^{-11}$$

The current uncertainty is well below the $\pm 60 \times 10^{-11}$ experimental error from E821.
<table>
<thead>
<tr>
<th># n of loops</th>
<th>$C_i \left[ (\alpha/\pi)^n \right]$</th>
<th>$a_{\mu}^{\text{QED}} \times 10^{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+0.5</td>
<td>116140973.289 (43)</td>
</tr>
<tr>
<td>2</td>
<td>+0.765 857 410(26)</td>
<td>413217.620 (14)</td>
</tr>
<tr>
<td>3</td>
<td>+24.050 512 28(46)</td>
<td>30141.905 (1)</td>
</tr>
<tr>
<td>4</td>
<td>+130.8105(85)</td>
<td>380.807 (25)</td>
</tr>
<tr>
<td>5</td>
<td>+663.0(20.0)</td>
<td>4.483 (137)</td>
</tr>
<tr>
<td>tot</td>
<td></td>
<td>116584718.104 (0.147)</td>
</tr>
</tbody>
</table>

1 diagram
Schwinger 1948

2 7 diagrams
Peterman 1957, Sommerfield 1957

3 72 diagrams
Lautrup, Peterman, de Rafael 1974, Laporta, Remiddi 1996

4 about 1000 diagrams
Kinoshita 1999, Kinoshita, Nio 2004

5 estimate of leading terms
Karshenboim 93, Czarnecki, Marciano 00, Kinoshita, Nio 05
Note on 3–loop contribution (Remiddi et al., Remiddi, Laporta 1996 [after 27 years]):

\[
A_{1\text{ uni}}^{(6)} = \frac{28259}{5184} + \frac{17101}{810} \pi^2 - \frac{298}{9} \pi^2 \ln 2 + \frac{139}{18} \zeta(3) + \frac{100}{3} \left\{ \text{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24} \ln^4 2 - \frac{1}{24} \pi^2 \ln^2 2 \right\}
- \frac{239}{2160} \pi^4 + \frac{83}{72} \pi^2 \zeta(3) - \frac{215}{24} \zeta(5) = 1.181\,241\,456\,587\ldots
\]

Result turned out to be surprisingly compact.
Note on 4–loop contribution (Kinoshita et al., Aoyama, Hayakawa, Kinoshita and Nio 2007):

4-loop Group V diagrams. 47 self-energy-like diagrams of $M_{01} - M_{47}$ represent 518 vertex diagrams [by inserting the external photon vertex on the virtual muon lines in all possible ways].

30 years of heroic effort $A_1^{(8)} = -1.9144(35)$ [error from Monte Carlo integration]. Recent, shift $(\alpha/\pi)^4$ coefficient by $0.19 \ (10\%) \ [7\sigma \ shift \ in \ a_e]$. Note that the universal $O(\alpha^4)$ contribution is sizable, about 6 standard deviations at current experimental accuracy, and a precise knowledge of this term is absolutely crucial for the comparison between theory and experiment.
Weak Contributions

W $\mu$ + $\nu\mu$ + $Z$ + $H$

Brodsky, Sullivan 67, ...
Bardeen, Gastmans, Lautrup 72
Higgs contribution tiny!

$a_{\mu}^{\text{weak(1)}} = (194.82 \pm 0.02) \times 10^{-11}$

Kukhto et al 92
potentially large terms $\sim G_F m_\mu^2 \alpha / \pi \ln \frac{M_Z}{m_\mu}$

Peris, Perrottet, de Rafael 95
quark-lepton (triangle anomaly) partial

Czarnecki, Krause, Marciano 96

Heinemeyer, Stöckinger, Weiglein 04, Gribouk, Czarnecki 05 final full 2–loop result known

$a_{\mu}^{\text{weak(2)}} = (-42.08 \pm 1.5[m_H, m_t] \pm 1.0[\text{had}]) \cdot 10^{-11}$

Most recent evaluations: improved hadronic part (beyond QPM)

$a_{\mu}^{\text{weak}} = (153.2 \pm 1.0[\text{had}] \pm 1.5[m_H, m_t, 3-\text{loop}]) \times 10^{-11}$

(Knecht, Peris, Perrottet, de Rafael 02, Czarnecki, Marciano, Vainshtein 02)
Hadronic Contributions

General problem in electroweak precision physics:
contributions from hadrons (quark loops) at low energy scales

Leptons

Quarks

(a) Hadronic vacuum polarization $O(\alpha^2), O(\alpha^3)$

(b) Hadronic light-by-light scattering $O(\alpha^3)$

(c) Hadronic effects in 2-loop EWRC $O(\alpha G_F m^2_\mu)$

Light quark loops $\rightarrow$ Hadronic “blob”
### Theory vs Experiment; do we see New Physics?

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value</th>
<th>Error</th>
<th>Reference</th>
</tr>
</thead>
</table>
| QED incl. 4-loops+LO 5-loops                      | 11 658 471.81 | 0.02   | Remiddi et al., Kinoshita et al. ...
| Leading hadronic vacuum polarization              | 690.3       | 5.3    | 2009 update                       |
| Subleading hadronic vacuum polarization           | -10.0       | 0.2    | 2006 update                       |
| Hadronic light–by–light                           | 11.6        | 3.9    | new evaluation (J&N)              |
| Weak incl. 2-loops                                | 15.32       | 0.22   | CMV06                            |

| Theory                                            | 11 659 179.0 | 6.5    | –                                 |
| Experiment                                        | 11 659 208.0 | 6.4    | BNL                               |
| The. - Exp.                                       | -29.0        | 9.0    | –                                 |

3.2 standard deviations

Standard model theory and experiment comparison [in units $10^{-10}$]
Progress in the Determination of the Hadronic Vacuum Polarization

**BaBar:** New: final $e^+e^- \rightarrow \pi\pi\gamma$ data Aug 2009 $\pi\pi$-spectrum from one experiment in large energy range !!!

- **Figure (a):** Data/QED as a function of $m_{\mu\mu}$.
- **Figure (b):** Cross section in nb as a function of $\sqrt{s'}$.
- **Figure (c):** Additional data and analysis.
Compilation of $\pi\pi$–data including new BaBar data from radiative return measurement at the $\Upsilon(4S)$ resonance (Davier et al 2009).
Relative cross section comparison between individual experiments Shown are BABAR (top left), KLOE (top right), CMD2 (bottom left) and SND (bottom right).
Relative comparison between the combined $\tau$ (dark shaded) and $e^+e^-$ spectral functions (light shaded), normalized to the $e^+e^-$ result.
Recent results for $a_\mu^{\text{SM}}$ (in units of $10^{-11}$), subtracted by the central value of the experimental average. The shaded vertical band indicates the experimental error.
New $R$ values at 2.6, 3.07 and 3.65 GeV form BESII at 3.5%!
About the hadronic light-by-light scattering contribution

Hadrons in $\langle 0 | T \{ A^\mu(x_1) A^\nu(x_2) A^\rho(x_3) A^\sigma(x_4) \} | 0 \rangle$

Key object full rank-four hadronic vacuum polarization tensor

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int d^4x_1 \, d^4x_2 \, d^4x_3 \, e^{i(q_1 x_1 + q_2 x_2 + q_3 x_3)}$$

$$\times \langle 0 | T \{ j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\rho(0) \} | 0 \rangle .$$

- non-perturbative physics
- general covariant decomposition involves 138 Lorentz structures of which
- 32 can contribute to $g - 2$
- fortunately, dominated by the pseudoscalar exchanges $\pi^0$, $\eta$, $\eta'$, ... described by the effective Wess-Zumino Lagrangian
- generally, pQCD useful to evaluate the short distance (S.D.) tail
- the dominant long distance (L.D.) part must be evaluated using some low energy effective model which includes the pseudoscalar Goldstone bosons as well as the vector mesons play key role

Hadronic light–by–light scattering is dominated by $\pi^0$ exchange in the odd parity channel, pion loops etc. at long distances (L.D.) and quark loops incl. hard gluonic corrections at short distances (S.D.)
The spectrum of invariant $\gamma\gamma$ masses obtained with the Crystal Ball detector. The three rather pronounced spikes seen are the $\gamma\gamma \rightarrow$ pseudoscalar (PS) $\rightarrow \gamma\gamma$ excitations: PS=$\pi^0, \eta, \eta'$

Note in pQCD with current quark masses one would get $a^{(6)}(lbl, u + d) = 8229.34 \cdot 10^{-11}$ and $a^{(6)}(lbl, s) = 17.22 \cdot 10^{-11}$ by adapting color, charge and mass in leptonic results. Meaning of these results more than questionable. Missing proper low energy structure of QCD. Need low energy effective theory mentioned before $\Rightarrow$ amount to calculate the following diagrams
Hadronic light–by–light scattering diagrams in a low energy effective model description. Diagrams (a) and (b) represent the long distance [L.D.] contributions at momenta \( p \leq \Lambda \), diagram (c) involving a quark loop yield the leading short distance [S.D.] part at momenta \( p \geq \Lambda \) with \( \Lambda \sim 1 \) to \( 2 \) GeV an UV cut-off

LD contribution requires low energy effective hadronic models:

**Resonance Lagrangian approach**

, incorporating VDM in accord with low energy structure of QCD (broken chiral symmetry, CHPT)

Based on such models, major efforts in estimating \( a_{\mu}^{\mathrm{LbL}} \) were made by

- (Hayakawa, Kinoshita, Sanda) (HKS 1995), (Hayakawa, Kinoshita) (HK 1998) [HLS model]
- (Bijnens, Pallante and Prades) (BPP 1995) [ENJL model]

Problem: matching L.D. with S.D. \( \Rightarrow \) results depend on matching cut off \( \Lambda \) \( \Rightarrow \) model dependence

(non-renormalizable low energy effective theory vs. renormalizable QCD)
Basic problem: \((s, s_1, s_2)\)–domain of \(\mathcal{F}_{\pi^0,\gamma^*\gamma^*}(s, s_1, s_2)\); here \((0, s_1, s_2)\)–plane

LbL 2D unbounded plane: \((s_1, s_2)\)-plane

??? Data, OPE, QCD factorization, Brodsky-Lepage approach

Novel approach: refer to quark–hadron duality of large-\(N_c\) QCD, hadrons spectrum known, infinite series of narrow spin 1 resonances (‘t Hooft 79) \(\Rightarrow\) no matching problem (resonance representation has to match quark level representation) (De Rafael 94, Knecht, Nyffeler 02)

- (Knecht, Nyffeler) (KN 2001) [LMD+V model] discovered a sign mistake in the \(\pi^0, \eta, \eta'\) exchange contribution, which changed the central value by \(+167 \cdot 10^{-11}\) at that time.

- (Melnikov, Vainshtein) (MV 2004) [LMD+V model] found additional inconsistencies in previous calculations, this time in the short distance constraints (QCD/OPE) used in matching the high energy behavior of the effective models used for the \(\pi^0, \eta, \eta'\) exchange contribution, shifts central value by \(+53 \cdot 10^{-11}\).
So far all used on-shell pion pole approximation!

- General form–factor $\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(s, s_1, s_2)$ is largely unknown

- The constant

$$
e^2 \mathcal{F}_{\pi^0 \gamma\gamma}(m_{\pi}^2, 0, 0) = \frac{e^2 N_c}{12\pi^2 f_{\pi}} = \frac{\alpha}{\pi f_{\pi}} \approx 0.025 \text{ GeV}^{-1}
$$

well determined by

$\pi^0 \rightarrow \gamma\gamma$ decay rate (from Wess-Zumino Lagrangian)

- Information on $\mathcal{F}_{\pi^0 \gamma^* \gamma}(m_{\pi}^2, -Q^2, 0)$ from $e^+e^- \rightarrow e^+e^-\pi^0$ experiments

![Diagram of e^+e^- → e^+e^-π^0 process](image)

**CELLO and CLEO measurement of the $\pi^0$ form factor $\mathcal{F}_{\pi^0 \gamma^* \gamma}(m_{\pi}^2, -Q^2, 0)$ at high space–like $Q^2$**
Brodsky–Lepage interpolating formula gives an acceptable fit.

\[ \mathcal{F}_{\pi^0 \gamma^* \gamma}(m_{\pi}^2, -Q^2, 0) \simeq \frac{1}{4\pi^2 f_{\pi}} \frac{1}{1 + \left(\frac{Q^2}{8\pi^2 f_{\pi}^2}\right)} \sim \frac{2f_{\pi}}{Q^2} \]

Assuming the pole approximation this FF has been used by all authors (HKS,BPP,KN) in the past, but has been criticized recently (MV and FJ07).

- Melnikov, Vainshtein: in chiral limit vertex with external photon must be non-dressed! i.e. use \( \mathcal{F}_{\pi^0 \gamma^* \gamma}(0, 0, 0) \) to avoid kinematic inconsistency, thus no VDM damping ⇒ result increases by 30%!

- In \( g - 2 \) external photon at zero momentum ⇒ only \( \mathcal{F}_{\pi^0 \gamma^* \gamma}(-Q^2, -Q^2, 0) \) is consistent with kinematics. Unfortunately, this off–shell form factor is not known and in fact not measurable and CELLO/CLEO constraint does not apply!

- Chiral limit in this case not a reasonable approximation!

Urgently need “model” for off–shell form–factor!
Evaluation of $a_{\mu}^{\text{LbL}}$ in the large-$N_c$ framework

- Knecht & Nyffeler and Melnikov & Vainshtein were using pion-pole approximation together with large-$N_c$ $\pi^0 \gamma \gamma$-formfactor
- FJ & A. Nyffeler: relax from pole approximation, using KN off-shell LDM+V formfactor

\[
\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(p_\pi^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{\mathcal{P}(q_1^2, q_2^2, p_\pi^2)}{Q(q_1^2, q_2^2)} \\
\mathcal{P}(q_1^2, q_2^2, p_\pi^2) = h_7 + h_6 p_\pi^2 + h_5 (q_2^2 + q_1^2) + h_4 p_\pi^4 + h_3 (q_2^2 + q_1^2) p_\pi^2 \\
+ h_2 q_1^2 q_2^2 + h_1 (q_2^2 + q_1^2)^2 + q_1^2 q_2^2 (p_\pi^2 + q_2^2 + q_1^2)) \\
Q(q_1^2, q_2^2) = (q_1^2 - M_1^2) (q_1^2 - M_2^2) (q_2^2 - M_1^2) (q_2^2 - M_2^2)
\]

all constants are constraint by SD expansion (OPE), except for $h_3 + h_- 4 = 2 c_{VT}$ with $c_{VT} = M_{V_1}^2 M_{V_2}^2 \chi/2$

and $\Pi_{VT}(0) = -\langle \bar{\psi}\psi \rangle_0/2 \chi$ with evaluations of $\chi[\text{GeV}^{-2}]$

$\chi[\text{GeV}^{-2}] = -2.7$ (Ball et al. '03) $-3.3$ (LMD) $-8.2$ (Ioffe&Smilga '84) $-8.9$ (Vainshtein '03)

First off-shell calculations:
FJ 08/Frascati: $h_6 = (-5 \pm 5 \text{ GeV}^4$ (positivity $\Rightarrow$ WZW bounded, QPM), stability
Nyffeler 09 : $h_6 = (+5 \pm 5 \text{ GeV}^4$ (LDM vs LDM+V smoothness)
My own calculation: $h_3 \in [-10, 10] \text{ GeV}^{-2}$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$a_\mu (\text{LbL}; X) \times 10^{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0, \eta, \eta'$</td>
<td>$93.91 \pm 12.40$</td>
</tr>
<tr>
<td>$a_1, f_1^f, f_1$</td>
<td>$28.13 \pm 5.63$</td>
</tr>
<tr>
<td>$a_0, f_0^f, f_0$</td>
<td>$-5.98 \pm 1.20$</td>
</tr>
</tbody>
</table>

JN09 based on Nyffeler 09:

\[
a_\mu^{\text{LbL;had}} = (116 \pm 39) \cdot 10^{-11}
\]

Summary of most recent results

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BPP</th>
<th>HKS</th>
<th>KN</th>
<th>MV</th>
<th>BP</th>
<th>PdRV</th>
<th>N/JN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0, \eta, \eta'$</td>
<td>$85 \pm 13$</td>
<td>$82.7 \pm 6.4$</td>
<td>$83 \pm 12$</td>
<td>$114 \pm 10$</td>
<td>$-$</td>
<td>$114 \pm 13$</td>
<td>$99 \pm 16$</td>
</tr>
<tr>
<td>$\pi, K$ loops</td>
<td>$-19 \pm 13$</td>
<td>$-4.5 \pm 8.1$</td>
<td>$-$</td>
<td>$0 \pm 10$</td>
<td>$-$</td>
<td>$-19 \pm 19$</td>
<td>$-19 \pm 13$</td>
</tr>
<tr>
<td>axial vectors</td>
<td>$2.5 \pm 1.0$</td>
<td>$1.7 \pm 1.7$</td>
<td>$-$</td>
<td>$22 \pm 5$</td>
<td>$-$</td>
<td>$15 \pm 10$</td>
<td>$22 \pm 5$</td>
</tr>
<tr>
<td>scalars</td>
<td>$-6.8 \pm 2.0$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-7 \pm 7$</td>
<td>$-7 \pm 2$</td>
</tr>
<tr>
<td>quark loops</td>
<td>$21 \pm 3$</td>
<td>$9.7 \pm 11.1$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$2.3$</td>
<td>$21 \pm 3$</td>
</tr>
<tr>
<td>total</td>
<td>$83 \pm 32$</td>
<td>$89.6 \pm 15.4$</td>
<td>$80 \pm 40$</td>
<td>$136 \pm 25$</td>
<td>$110 \pm 40$</td>
<td>$105 \pm 26$</td>
<td>$116 \pm 39$</td>
</tr>
</tbody>
</table>
Note: MV and KN utilize the same model LMD+V form factor:

\[
F_{\pi \gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{4\pi^2 F^2_\pi}{N_c} \frac{q_1^2 q_2^2 (q_1^2 + q_2^2) - h_2 q_1^2 q_2^2 + h_5 (q_1^2 + q_2^2) + \left(N_c M_1^4 M_2^4/4\pi^2 F^2_\pi\right)}{(q_1^2 + M_1^2)(q_1^2 + M_2^2)(q_2^2 + M_1^2)(q_2^2 + M_2^2)},
\]

where \(M_1 = 769\) MeV, \(M_2 = 1465\) MeV, \(h_5 = 6.93\) GeV\(^4\).

with two modifications:

- form factor \(F_{\pi^0\gamma^*\gamma^*}(q_2^2, q_2^2, 0) = 1\): undressed soft photon (non-renormalization of ABJ) Note: to have anomaly correct does not imply that there is no damping! \(PVV\) anomaly quark loop is counter example; it has correct \(\pi\gamma\gamma\) in chiral limit (anomaly) and goes like \(1/q_i^4\) up to logs in all directions
- \(F_{\pi^0\gamma^*\gamma^*}(q_2^2, q_1^2, q_3^2) \sim F_{\pi^0\gamma^*\gamma^*}(m_{\pi}^2, q_1^2, q_3^2) = KN\)
  with \(h_2 = 0 \pm 20\) GeV\(^2\) (KN) vs. \(h_2 = -10\) GeV\(^2\) (MV) fixed by twist 4 in OPE \((1/q^4)\)
- \(a_1[f_1, f_1^*]\) different mixing scheme

Criticism: KN Ansatz only covers \((0, q_1^2, q_2^2)\)-plane, with consistent kinematics depends on 3 variables \(\rightarrow 2\)-dim integral representation no longer valid.

Is this the final answer? How to improve? A limitation to more precise \(g - 2\) tests?

Looking for new ideas to get ride of model dependence

- Lattice QCD will provide an answer [far future (“yellow” region only)]!
A new representation for single particle exchange in LbL

- $a_\mu$ does not depend on direction of muon momentum $p \Rightarrow$ may average in Euclidean space over the directions $\hat{P}$:

$$\langle \cdots \rangle = \frac{1}{2\pi^2} \int d\Omega(\hat{P}) \cdots$$

Hadronic single particle exchange amplitudes independent of $p \Rightarrow$ 2 integrations may be done analytically: amplitudes $T_i$, propagators (4) $\equiv (P + Q_1)^2 + m_\mu^2$ and (5) $\equiv (P - Q_2)^2 + m_\mu^2$ with $P^2 = -m_\mu^2$

$$\langle \frac{1}{(4)} \frac{1}{(5)} \rangle = \frac{1}{m_\mu^2 R_{12}} \arctan \left( \frac{zx}{1 - zt} \right)$$

$$\langle (P \cdot Q_1) \frac{1}{(5)} \rangle = -(Q_1 \cdot Q_2) \frac{(1 - R_{m2})^2}{8m_\mu^2}$$
\[
\langle (P \cdot Q_2) \frac{1}{(4)} \rangle = (Q_1 \cdot Q_2) \frac{(1 - R_{m1})^2}{8m_\mu^2} \\
\langle \frac{1}{(4)} \rangle = -\frac{1 - R_{m1}}{2m_\mu^2} \\
\langle \frac{1}{(5)} \rangle = -\frac{1 - R_{m2}}{2m_\mu^2}
\]

\[R_{mi} = \sqrt{1 + 4m_\mu^2/Q_i^2}, \quad (Q_1 \cdot Q_2) = Q_1 Q_2 t, \quad t = \cos \theta, \quad \theta = \text{angle between } Q_1 \text{ and } Q_2.\]

Denoting \(x = \sqrt{1 - t^2}\), we have \(R_{12} = Q_1 Q_2 x\) and

\[z = \frac{Q_1 Q_2}{4m_\mu^2} (1 - R_{m1}) (1 - R_{m2}).\]

- For any hadronic form-factor end up with 3–dimensional integral over \(Q_1 = |Q_1|, Q_2 = |Q_2| \) and \(t = \cos \theta:\)

\[a_\mu(\text{LbL}; \pi^0) = -\frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt \sqrt{1 - t^2} \frac{Q_1^3 Q_2^3}{Q_1^3 Q_2^3} \times (F_1 P_6 I_1(Q_1, Q_2, t) + F_2 P_7 I_2(Q_1, Q_2, t))\]
where $P_6 = 1/(Q_2^2 + m^2_\pi)$, and $P_7 = 1/(Q_3^2 + m^2_\pi)$ denote the Euclidean single particle exchange propagators. $I_1$ and $I_2$ known integration kernels. The non-perturbative factors are

$$F_1 = \mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_2^2, q_1^2, q_3^2) \mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_2^2, q_2^2, 0),$$
$$F_2 = \mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_3^2, q_1^2, q_2^2) \mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_3^2, q_3^2, 0).$$

Note: SU(3) flavor decomposition of em current $\rightarrow$ weight factors

$$W^{(a)} = \frac{(\text{Tr} [\lambda_a \hat{Q}^2])^2}{\text{Tr} [\lambda_a^2] \text{Tr} [\hat{Q}^4]}; \quad W^{(3)} = \frac{1}{4}, \quad W^{(8)} = \frac{1}{12}, \quad W^{(0)} = \frac{2}{3}.$$ 

where $\text{Tr} [\hat{Q}^4] = 2/9$ is the overall normalization such that $\sum_a W^{(a)} = 1$. Note $(W^{(8)} + W^{(0)})/W^{(3)} = 3$, higher states enhanced in coupling by factor 3!

[Melnikov&Vainshtein] overlooked by previous analyses [HKS,HK,BPP].
Summary and Outlook

\[ a_{\mu}^{\text{exp}} = 1.16592080(63) \times 10^{-3} \quad a_{\mu}^{\text{the}} = 1.16591790(65) \times 10^{-3} \]

\[ \delta a_{\mu}^{\text{NP?}} = a_{\mu}^{\text{exp}} - a_{\mu}^{\text{the}} = (290 \pm 90) \cdot 10^{-11}, \]

\[ \Rightarrow 3.2 \sigma \]

Uncertainties:

<table>
<thead>
<tr>
<th>Source</th>
<th>Value</th>
<th>Uncertainties</th>
</tr>
</thead>
<tbody>
<tr>
<td>experiment</td>
<td>0.54ppm</td>
<td>6.3 \times 10^{-10}</td>
</tr>
<tr>
<td>theory</td>
<td>0.57ppm</td>
<td>6.5 \times 10^{-10}</td>
</tr>
<tr>
<td>new BNL 969 prop.</td>
<td>0.20ppm</td>
<td>2.4 \times 10^{-10}</td>
</tr>
</tbody>
</table>
New physics sensitivity: (example)

\[ \Delta a_\mu^{\text{SUSY}} / a_\mu \simeq 1.25 \text{ppm} \left( \frac{100 \text{GeV}}{\tilde{m}} \right)^2 \tan \beta \]

\( \tilde{m} \) lightest SUSY particle; SUSY requires two Higgs doublets

\[ \tan \beta = \frac{v_1}{v_2}, v_i = \langle H_i \rangle ; \ i = 1, 2 \]

\[ \tan \beta \sim m_t / m_b \sim 40 \ [4 - 40] \]

The supersymmetric contributions to \( a_\mu \) stem from

smuon–neutralino and sneutrino-chargino loops
New physics typically:

\[ a_{\mu}^{NP} = C \frac{m_{\mu}^2}{M_{NP}^2} \]

where naturally \( C = O(\alpha/\pi) \) (\( \sim \) lowest order \( a_{\mu}^{SM} \));

Typical New Physics scales required to satisfy \( \Delta a_{\mu}^{NP} = \delta a_{\mu} \):

<table>
<thead>
<tr>
<th>( C )</th>
<th>1</th>
<th>( \alpha/\pi )</th>
<th>( (\alpha/\pi)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{NP} )</td>
<td>2.0^{+0.4}_{-0.3} \text{ TeV}</td>
<td>100^{+21}_{-13} \text{ GeV}</td>
<td>5^{+1}_{-1} \text{ GeV}</td>
</tr>
</tbody>
</table>
Given theory results only differ by $a_\mu^{\text{had}}(1)$. 

<table>
<thead>
<tr>
<th>Theory</th>
<th>Result</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^+e^-$</td>
<td>181.3 ± 16</td>
<td>1.6σ</td>
</tr>
<tr>
<td>$e^+e^-$ (H.O. had)</td>
<td>180.5 ± 5.6</td>
<td>3.3σ</td>
</tr>
<tr>
<td>$e^+e^-$ (L.O. had)</td>
<td>180.4 ± 5.1</td>
<td>3.4σ</td>
</tr>
<tr>
<td>$\tau-e^+e^-$ (H.O. had)</td>
<td>193.2 ± 5.2</td>
<td>1.8σ</td>
</tr>
<tr>
<td>$\tau-e^+e^-$ (L.O. had)</td>
<td>183.4 ± 4.9</td>
<td>3.1σ</td>
</tr>
</tbody>
</table>

$\mu \times 10^{10} - 11659000$
The Muon $g - 2$

"the closer you look the more there is to see"

History of sensitivity to various contributions

The anomalous magnetic moment of the muon by itself a tiny $0.116\%$ effect now measured at $5 \cdot 10^{-7}$!
This was Muon to the Deepest

Thanks for your attention! Dziękuję!

Book: F. Jegerlehner,
The Anomalous Magnetic Moment of the Muon,
Springer Tracts in Modern Physics,
Vol. 226, November 2007
Backup slides: The $\tau$ vs. $e^+e^-$ problem

Additional data: $\tau$–data + CVC

$\pi^+\pi^-, \ldots [I = 1]$
ALEPH–Coll., (OPAL, CLEO), Alemany, Davier, Höcker 1996, 
Belle–Coll. Fujikawa, Hayashii, Eidelman 2008

\[ \tau^- \rightarrow X^- \nu_\tau \leftrightarrow e^+ e^- \rightarrow X^0 \]

where \( X^- \) and \( X^0 \) are hadronic states related by isospin rotation. The \( e^+ e^- \)
cross-section is then given by

\[ \sigma_{e^+e^- \rightarrow X^0} = \frac{4\pi\alpha^2}{s} v_{1,X^-}, \sqrt{s} \leq M_\tau \]

in terms of the \( \tau \) spectral function \( v_1 \).

- mainly improves the knowledge of the \( \pi^+ \pi^- \) channel (\( \rho \)–resonance contribution)
- which is dominating in \( a_\mu^{\text{had}} \) (72%)

\[
I = 1 \sim 75\% ; I = 0 \sim 25\%
\]

\( \tau \)–data cannot replace \( e^+ e^- \)–data

\[
\delta a_\mu : 15.6 \times 10^{-10} \rightarrow 10.2 \times 10^{-10}
\]

\[
\delta \Delta \alpha : 0.00067 \rightarrow 0.00065 \quad (ADH1997)
\]
Most recent measurement from Belle (2008):

\[ |F_\pi|^2 \]

\[
\begin{align*}
(M_{\pi})^2 & = (\text{GeV/c}^2)^2 \\
(0, 0.5, 1, 1.5, 2, 2.5, 3) & \text{ GeV/c}^2
\end{align*}
\]

Belle
ALEPH
CLEO
\( \rho(770) + \rho(1450) + \rho(1700) \)

\( \text{G&F Fit} \)
$e^+e^-$–data$^\ast$ = data corrected for isospin violations:

In $e^+e^-$ (neutral channel) $\rho - \omega$ mixing due isospin violation be quark mass difference

$$m_u \neq m_d \Rightarrow$$

l=0 component; to be subtracted for comparison with $\tau$ data

$$|F(s)|^2 = \left( |F(s)|^2 - \text{data} \right) \div \left( 1 + \frac{\epsilon s}{s_{\omega} - s} \right)$$

with $s_{\omega} = (M_{\omega} - i \frac{1}{2} \Gamma_{\omega})^2$

$\epsilon$ determined by fit to the data: $\epsilon = 0.00172$

CMD-2 data for $|F_\pi|^2$ in $\rho - \omega$ region together with Gounaris-Sakurai fit. Left before subtraction right after subtraction of the $\omega$.

l=0 component to be added to $\tau$ data for calculating $a_{\mu}^{\text{had}}$!
Other isospin-breaking corrections (Cirigliano et al. 2002, López Castro et al. 2007)

Left: Isospin-breaking corrections $G_{EM}$, FSR, $\beta_0^3(s)/\beta^-_0(s)$ and $|F_0(s)/F_-(s)|^2$.

Right: Isospin-breaking corrections in $I = 1$ part of ratio $|F_0(s)/F_-(s)|^2$: π mass splitting $\delta m_\pi = m_{\pi^\pm} - m_{\pi^0}$, ρ mass splitting $\delta m_\rho = m_{\rho^\pm} - m_{\rho_{bare}^0}$, and ρ width splitting $\delta \Gamma_\rho$. 
New isospin corrections applied shift in mass and width [as advocated by S. Ghozzi and FJ in 2003!!] plus changes [López Castro, Toledo Sánchez et al 2007] below the $\rho$ which Davier et al say are not understood! The discrepancy now substantially reduced but with the KLOE data persists.

$e^+e^- \text{ vs } \tau$ spectral functions: $|F_{ee}|^2/|F_\tau|^2 - 1$ as a function of $s$. Isospin-breaking (IB) corrections are applied to $\tau$ data with its uncertainties included in the error band.
The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$. 

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$. 

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$. 

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$.

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$. 

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$. 

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$. 

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$. 

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$. 

The measured branching fractions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ compared to the predictions from the $e^+ e^- \rightarrow \pi^+ \pi^-$ spectral functions (after isospin-breaking corrections). (Named $e^+ e^-$ results for $0.63 - 0.958\text{GeV}$). The long and short vertical error bands correspond to the $\tau$ and $e^+ e^-$ averages of $25.42 \pm 0.10$ and $24.76 \pm 0.25$, respectively.

Note -2% in Belle $\tau$ data means $25.42 \rightarrow 24.91$ in agreement with $e^+ e^- \left[ |F_\tau(0)|^2 = 1.02 \rightarrow |F_\tau(0)|^2 = 1 \right]$.
Possible origin of problems:
• Unknown isospin violations in parameters: \( m_{\rho^+} - m_{\rho^0}, m_{\rho'^+} - m_{\rho'_0}, m_{\rho''+} - m_{\rho''_0} \); same for widths, mixing parameters; largely not established (theor. and exper.)

Cottingham formula calculating \( m_{\pi^-}^2 - m_{\pi^0}^2 \) very successfully suggests \( \Delta m_{\rho} = \Delta m_{\pi} \Rightarrow m_{\rho^+} - m_{\rho^0} \simeq 0.81 \text{ MeV} \sim 1 \text{ MeV} \)

Also: \( \Gamma_{\rho^0} = \left( \frac{m_{\rho^0}}{m_{\rho^-}} \right)^3 \left( \frac{\beta^0}{\beta^-} \right)^3 \Gamma_{\rho^-} + \Delta \Gamma_{\text{em}} \Rightarrow \Gamma_{\rho^-} - \Gamma_{\rho^0} \simeq 2.1 \pm 0.5 \text{ MeV radiative em corrections now included} \)

• Needed what is measured in \( e^+ e^- \): \( |A_{I=1}(s) + A_{I=0}(s)|^2 < |A_{I=1}(s)|^2 + |A_{I=0}(s)|^2 \);
• \( \tau \) evaluations based on \( |A_{I=1}(s)|^2 + |A_{I=0}(e^+ e^-}(s)|^2 \) which may overestimate the effects; separation of \( |A_{I=0}(e^+ e^-}(s)|^2 \) using Gounaris-Sakurai fit of the \( \rho - \omega \) \[ \imath_{\rho\omega} = (2.02 \pm 0.1) \times 10^{-3} \]; (see HLS model calculation presented by Benayoun et al. which claims large diminution by interference).
• hadronic final state photon radiation not under quantitative control, in \( \tau \)-decay enhanced short distance sensitivity (UV-log modeled by quark parton model, rest by sQED)
\( \tau \) data vs. residual distribution in the fit of \( \tau \) data: Left: BELLE+CLEO, Right: ALEPH+BELLE+CLEO (from Benayoun et al 09))

BELLE: best fit of \( |F_\tau(s)|^2 \) yields \( F_\tau(0) = 1.02 \pm 0.01 \pm 0.04 \Rightarrow \) this violates em current conservation.

Benayoun et al. 2009 suggest that normalization may be wrong \( \rightarrow \) shift down data by 2%; actually with global shift by -4.5% perfect agreement with Novosibirsk \( e^+e^- \) data (as a distribution). Is the main problem that ALEPH lies very high ???